Lecture 3

Method of Weighted Residuals (MWR)

Today

- We have so far approximated functions by writing them as either
  - Global polynomials with unknown coefficients
  - Piecsewise polynomials with unknown coefficients

  and finding the coefficients by equating the function and expansion at a set of interpolation (collocation) points
- Now we will look at the Method of Weighted Residuals, whereby we will find the coefficients by minimizing the error over the (global) interval

Least Squares Broken Line Approximation (not an interpolation)

Recall:

$$I_2\ f \equiv \sum_{i=1}^{n} f(x_i)\Lambda_i(x)$$

Find:

$$L_2\ f \equiv \sum_{i=1}^{n} \alpha_i \Lambda_i(x)$$

such that

$$\int_a^b |f(x) - L_2\ f|^2\, dx$$

is minimized.

Proceeding to minimize the LHS:

$$\min \int \left| f(x) - \sum_{i=1}^{n} \alpha_i \Lambda_i(x) \right|^2 dx = F(\alpha_i)$$

Set

$$\frac{\partial F}{\partial \alpha_j} = 0 = \int_a^b 2 \left(f(x) - \sum_{i=1}^{n} \alpha_i \Lambda_i(x) \right) (-\Lambda_j(x))\, dx$$

And we arrive at:

$$\sum_{j=1}^{n} M_{ij}\alpha_j = \beta_i$$

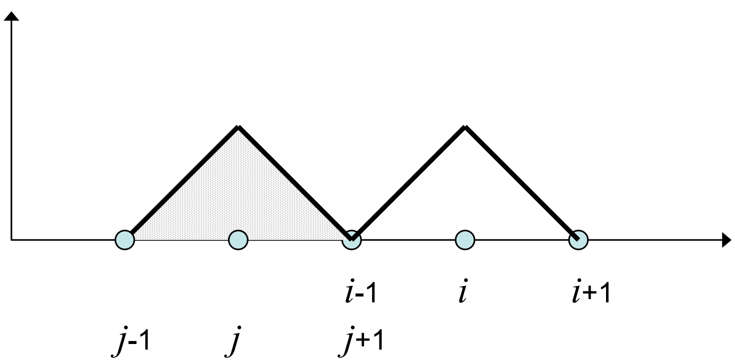
where

$$M_{ij} = \left(\int_a^b \Lambda_i(x)\Lambda_j(x) dx \right) = M_{ji} \qquad \beta_i = \int_a^b \Lambda_i(x) f(x) dx$$

This is a linear system of equations. You can easily see that the 2nd derivatives are positive and we therefore have a minimum.

Some details:

$$M_{ij} = \left(\int_a^b \Lambda_i(x) \Lambda_j(x) dx \right) = \begin{cases} \frac{x_i - x_{i-1}}{6} & j = i-1 \text{ (lower diagonal)} \\ \frac{x_{i+1} - x_{i-1}}{3} & j = i \text{ (diagonal)} \\ \frac{x_{i+1} - x_i}{6} & j = i+1 \text{ (upper diagonal)} \\ 0 & \text{otherwise} \end{cases}$$



Some details (take $i \neq 1$ and $i \neq n$).

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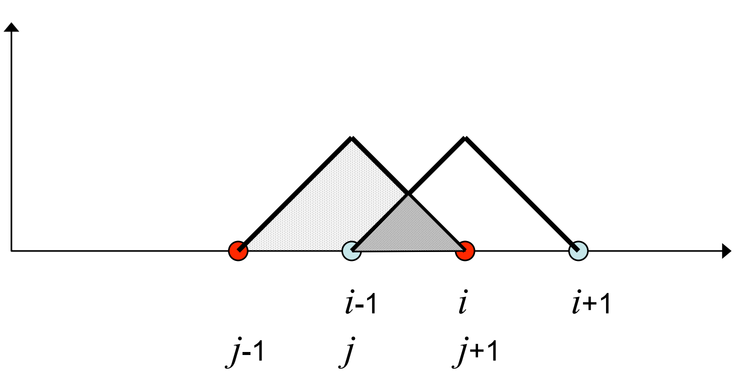

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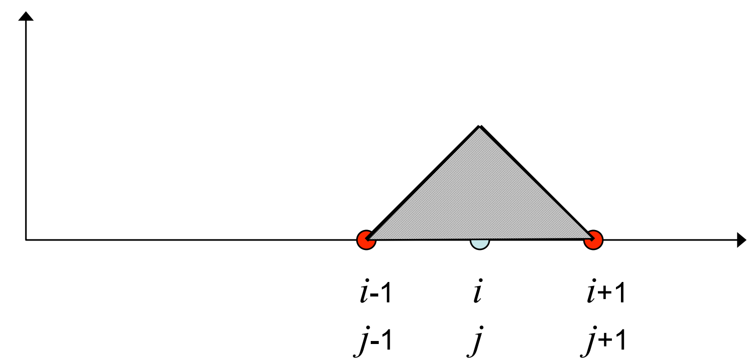

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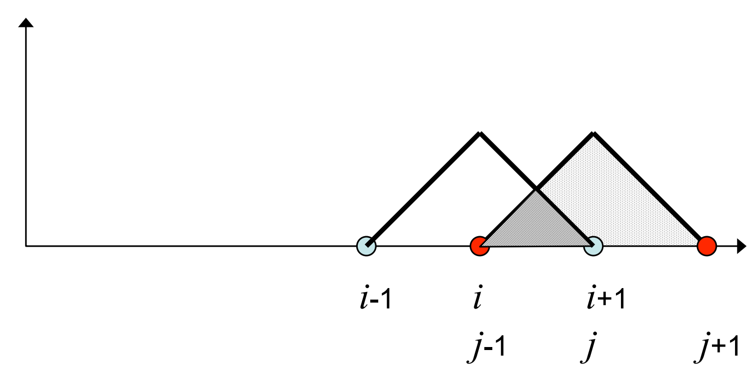
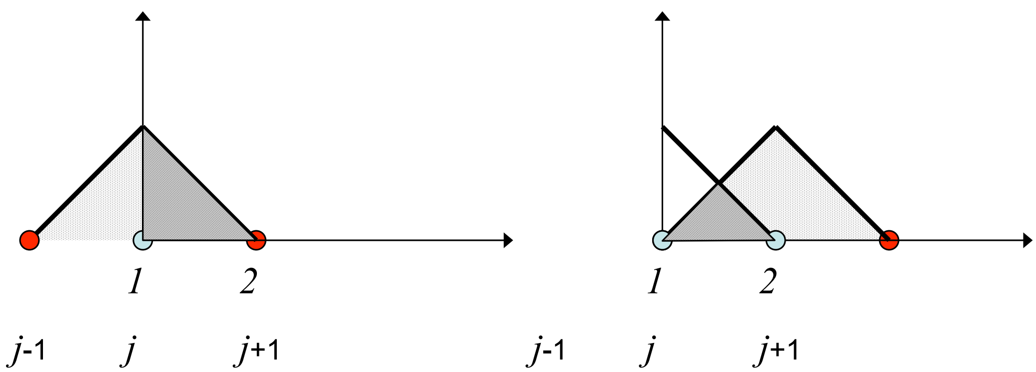


Some details $i = 1$:

$$M_{1j} = \begin{cases} \frac{x_2 - x_1}{3} & j = 1 \\ \frac{x_2 - x_1}{6} & j = 2 \\ 0 & \text{otherwise} \end{cases}$$



To simplify notation, let:

$$\Delta_j = x_{j+1} - x_j$$

In matrix form:

$$\mathbf{M} = \begin{pmatrix} \frac{\Delta_1}{3} & \frac{\Delta_1}{6} & & & & & \\ \frac{\Delta_1}{6} & \frac{\Delta_2+\Delta_1}{3} & \frac{\Delta_2}{6} & & & & \\ & \cdots & \cdots & \cdots & & & \\ & & \frac{\Delta_{j-1}}{6} & \frac{\Delta_j+\Delta_{j-1}}{3} & \frac{\Delta_j}{6} & & \\ & & & \cdots & \cdots & \cdots & \\ & & & & \frac{\Delta_{n-2}}{6} & \frac{\Delta_{n-1}+\Delta_{n-2}}{3} & \frac{\Delta_{n-1}}{6} \\ & & & & & \frac{\Delta_{n-1}}{6} & \frac{\Delta_{n-1}}{3} \end{pmatrix}$$

This is a tridiagonal matrix and is solved by Gauss elimination in $O(n)$ operations. Pivoting is not needed if there is "diagonal dominance", $|b_j| > |a_j| + |c_j|$, where a, b, and c are the lower diagonal, diagonal, and upper diagonal.

MWR more generally

$$P^n f(x) = \sum_{j=1}^{n} \alpha_j \phi_j(x), \quad x \in [a, b].$$

Eq. 3-1

The functions ϕ_j are called basis or *trial functions*.
Define the inner product:

$$\langle f(x),\ g(x)\rangle_w = \int_a^b f(x)g(x)w(x)dx$$

Eq. 3-2

where $w(x)$ is an appropriate weight function.

Take inner product of expansion with a set of *test functions*, $\psi_k(x)$ (i.e. project the expansion on to the space of functions ψ_j)

$$\langle P^n f(x),\ \psi_k(x)\rangle_w = \langle \sum_{j=1}^{n} \alpha_j \phi_j(x),\ \psi_k(x)\rangle_w = \sum_{j=1}^{n} \alpha_j \langle \phi_j,\ \psi_k\rangle_w$$

Or, in matrix notation:

$$\beta = M\alpha$$

Eq. 3-3

where

$$\beta_i = \langle P^n f(x),\ \psi_i(x)\rangle_w, \qquad M_{ij} = \langle \phi_j,\ \psi_i\rangle_w.$$

Special Cases

1. Least-squared, also called Galerkin, let

$$\psi_k(x) = \phi_k(x), \forall k$$

Eq. 3-4

2. Interpolation, also called collocation, let

$$\psi_k(x) = \delta(x - x_k)$$

Eq. 3-5

where $x_k, k = 1, 2, \cdots n$ are the *interpolation points*.

3. Some particularly useful Galerkin projections use functions that are orthogonal:

$$\langle \phi_j,\ \phi_i\rangle_w = \delta_{ij}$$

Boris Grigorievich Galerkin
B 1871 Belarus
D 1945 USSR

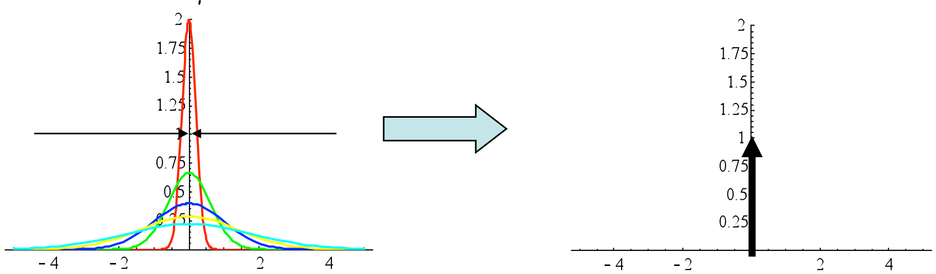
Aside: Dirac "delta function"

- Important in numerical analysis
- Read Chapter 5 of Bracewell
- Generalized function (distribution)
- Only integral properties are important (not details of shape)
- Consider as limiting case of a sequences of functions (non-unique) with:

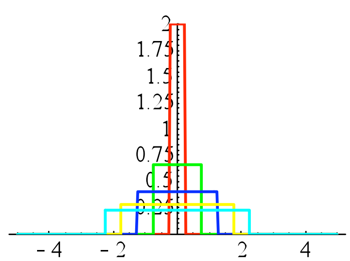
$$\int_{-\infty}^{\infty} \delta(x)dx = \lim_{\tau \to 0} \int_{-\infty}^{\infty} p_\tau(x)dx = 1$$

Eq. 3-6

$$p_\tau(x) = \frac{1}{\tau} e^{-\pi x^2/\tau^2}$$

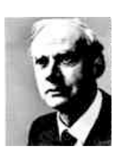


- Another possible sequence



$$p_\tau(x) = \frac{1}{\tau}\Pi(\frac{x}{\tau})$$

Paul Adrien Maurice Dirac
B 1902 England
D 1984 USA
I was taught at school never to start a sentence without knowing the end of it



Ae232a. © Tim Colonius

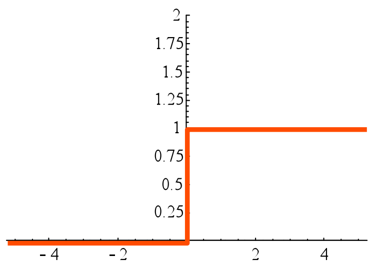
- Key properties:

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = \lim_{\tau \to 0} \int_{-\infty}^{\infty} p_\tau(x-a) f(x) dx = f(a)$$ Eq. 3-6

$$\int_{-\infty}^{\infty} \delta'(x-a) f(x) dx = \lim_{\tau \to 0} \int_{-\infty}^{\infty} f(x) \frac{d}{dx}(p_\tau(x-a)) f(x) dx = -f'(a)$$ Eq. 3-7

(these assume that $f(x)$ and $f'(x)$ are continuous at $x = a$)

- Related Heaviside function:



$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

$$H'(x) = \delta(x)$$

Note that when $f(x)$ is not continuous at $x = a$, then we have:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a) dx = \frac{1}{2}\left(f(a+) + f(a-)\right)$$ Eq. 3-8

Two other useful properties:

1. $f(x)\delta(x-a) = f(a)\delta(x-a)$ Eq. 3-9
2. $\delta(ax+b) = \frac{1}{|a|}\delta(x+\frac{b}{a})$ Eq. 3-10

…back to MWR

- Example 1 – Broken Line

Let

$$a = x_1 < x_2 < \ldots < x_n = b$$

and let

$$\Lambda_i(x) \equiv \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & \text{if } x_{i-1} \le x < x_i \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & \text{if } x_i \le x < x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

1. $w(x) = 1$
2. Galerkin ($P \to L2$)

$$\phi_j = \Lambda_j \qquad \psi_j = \Lambda_j$$

3. Collocation ($P \to I2$)

$$I_2 f(x) = \sum_{j=1}^{n} \alpha_j \Lambda_j(x)$$

$$\phi_j = \Lambda_j \qquad \psi_j = \delta(x - x_j)$$

$$\langle f(x),\ \delta(x - x_j)\rangle_w = f(x_j)$$

$$\langle \Lambda(x_j),\ \delta(x - x_k)\rangle_w = \Lambda_j(x_k) = \delta_{jk}$$

$$f(x_k) = \sum_{j=1}^{n} \alpha_j \delta_{jk} = \alpha_k$$

- Example 2

MWR Galerkin for $f(x) = u^2(x)$.

$$Pf(x) = \sum_j \alpha_j \phi_j(x) \qquad Pu(x) = \sum_j \gamma_j \phi_j(x)$$

As before,

$$M\alpha = \beta \quad \text{where} \quad M_{ij} = \langle \phi_j,\ \phi_i\rangle_w \quad \beta_j = \langle f(x),\ \phi_j\rangle_w$$

and

$$\beta_k = \left\langle \left(\sum_j \gamma_j \phi_j(x) \right)^2,\ \phi_k \right\rangle_w \qquad \forall k$$

$$\beta_k = \sum_j \sum_m \gamma_j \gamma_m \underbrace{\langle \phi_j \phi_m, \phi_k \rangle_w}_{N_{kjm}} = \sum_j \sum_m \gamma_j \gamma_k N_{kjm}$$

$$\beta_k = \sum_j \sum_m \gamma_j \gamma_k N_{kjm}$$

- Will usually have simple enough basis and trial functions so that N can be computed analytically.
- Discrete convolution-like sum
- Requires n^2 operations
- If β known, n nonlinear equations to solve for γ

Summary

- Interpolation and least-squared function approximation are special cases of a general *projection* procedure called MWR
- We have not proved that anything about how $P^n f(x)$ converges to $f(x)$ in general (nor can much be proved without some specifics of test and trial functions).
- Projections of quadratic nonlinearity leads to a convolution-type sum